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Algebraic limit cycles in polynomial systems of differential equations*

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Abstract
Using elementary tools we construct cubic polynomial systems of differential equations with algebraic limit cycles of degrees 4, 5 and 6. We also construct a cubic polynomial system of differential equations having an algebraic homoclinic loop of degree 3. Moreover, we show that there are polynomial systems of differential equations of arbitrary degree that have algebraic limit cycles of degree 3, as well as give an example of a cubic polynomial system of differential equations with two algebraic limit cycles of degree 4.

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1. Introduction

By definition, a *planar polynomial system of differential equations* is a system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = P(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y. The *degree* i of the polynomial system of differential equations is the maximum of the degrees of the polynomials P and Q. In what follows, a planar polynomial system of differential equations of degree 2 or 3 will be called simply a *quadratic* or a *cubic* system, respectively.

In 1900, Hilbert [16] in the second part of its 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial systems of differential equations of a given degree, and also to study their distribution or configuration

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in the plane (see [23]). This has been one of the main problems in the qualitative theory of planar polynomial systems of differential equations in the 20th century. The contributions of Bamon [3] for the particular case of quadratic polynomial systems of differential equations, and mainly of Écalle [11] and Ilyashenko [18] proving that any polynomial vector field has finitely many limit cycles, have been the best results in this area. But until now the existence of the uniform upper bound is not proved. This problem remains open even for the quadratic systems.

Let $f \in \mathbb{R}[x, y]$, i.e. f is a polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of the polynomial system of differential equations (1) if for some polynomial $K \in \mathbb{R}[x, y]$, we have

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree i, then any cofactor has at most degree $i - 1$. Of course the curve $f = 0$ is formed by trajectories of the system (1). This justifies the name of the invariant algebraic curve. We define the *degree* of the invariant algebraic curve $f = 0$ as the degree of the polynomial f.

We recall that a *limit cycle* of a polynomial system of differential equations is an isolated periodic orbit in the set of all periodic orbits of the system. An *algebraic limit cycle of degree* k is an oval of an irreducible invariant algebraic curve $f(x, y) = 0$ of degree k which is a limit cycle of the system.

In this paper, we are mainly interested in studying the algebraic limit cycles of polynomial systems of differential equations of degree $\geqslant 3$. First, we recall what is known and unknown on the algebraic limit cycles of quadratic systems, and then we shall present our results on the algebraic limit cycles of polynomial differential systems.

In 1958, Qin [25] (see also [29]) proved that quadratic systems can have algebraic limit cycles of degree 2, moreover when such a limit cycle exists then it is the unique limit cycle of the system. Evdokimenco in [12–14] proved that quadratic systems do not have algebraic limit cycles of degree 3, for two different shorter proofs see [6, 22].

The first class of algebraic limit cycles of degree 4 was given in 1966 by Yablonskii [28]. The second class was found in 1973 by Filiptsov [15]. Recently, two new classes have been found and in [8] the authors proved that there are no other algebraic limit cycles of degree 4 for quadratic systems, see theorem 3. The uniqueness of these limit cycles has been proved in [5].

Applying convenient birational transformation to quadratic systems having algebraic limit cycles of degree 4 in [8], the authors obtained algebraic limit cycle of degrees 5 and 6 for quadratic systems, see theorem 4.

Open question 1. *In quadratic systems remains the following open questions related with algebraic limit cycles, see for instance [22].*

(i) What is the maximum degree of an algebraic limit cycle of a quadratic system?
(ii) Does there exist a chain of rational transformations which give examples of quadratic systems with algebraic limit cycles of arbitrary degree?
(iii) The maximal number of algebraic limit cycles that a quadratic system can have is at most 1?

In this paper, we study the algebraic limit cycles of polynomial systems of differential equations. It is known that there are cubic systems having algebraic limit cycles of degrees 2 and 3, see the beginning of section 2. In propositions 6, 7 and 8, we provide cubic systems having algebraic limit cycles of degrees 4, 5 and 6, respectively. These propositions are the main results of section 2.

In section 3, we study the algebraic ovals of degree 3 which will be used in section 4 for the characterization of all algebraic limit cycles of degree 3 regular at infinity (for more details, see section 4 and theorem 12). In section 4, we prove that any oval of an irreducible algebraic curve $f = 0$ of degree 3 without points satisfying $f = \partial f/\partial x = \partial f/\partial y = 0$ is a hyperbolic algebraic limit cycle of a convenient polynomial system of differential equations of degree $i \geqslant 3$, see theorem 15.

In section 4, we also provide some results on algebraic limit cycles of degree k, see theorem 14 and remark 16.

In section 5, we study the algebraic homoclinic loops of degree 3 which appear in cubic systems, see theorem 17 and proposition 18.

Finally, in section 6, we provide a cubic system having two algebraic limit cycles.

Open question 2. *In cubic systems there remains the following open questions related with algebraic limit cycles.*

(i) What is the maximum degree of an algebraic limit cycle of a cubic system?

(ii) Does there exist a chain of rational transformations which give examples of cubic systems with algebraic limit cycles of arbitrary degree?

(iii) The maximal number of algebraic limit cycles that a cubic system can have is at most 2?

2. Algebraic limit cycles of degree 2, 3, 4, 5 and 6

In this section, we present cubic systems having algebraic limit cycles. A well-known cubic system having the algebraic limit cycle $x^2 + y^2 = 1$ of degree 2 is

$$\dot{x} = -y + x(x^2 + y^2 - 1), \qquad \dot{y} = x + y(x^2 + y^2 - 1).$$

More examples of cubic systems having algebraic limit cycles of degree 2 can be found in [4, 20, 26].

Paper [21] is concerned with the cubic systems that have an invariant cubic curve of the form $y^2 = ax^3 + bx^2 + cx + d$. The author provides explicit examples of cubic systems having the bounded oval of the cubic curve $y^2 = x^3 + cx^2 + 1$ with $c < -3\sqrt[3]{2}/2$ as an algebraic limit cycle of degree 3. It turns out that there are cubic systems where this limit cycle is unique and others where the system has additionally non-algebraic limit cycles. We also discuss algebraic limit cycles of degree 3 for cubic systems in section 4.

In what follows, we give some examples of cubic systems having algebraic limit cycles of degree 4, 5, 6, respectively. The main idea is to apply a Poincaré transformation to known quadratic systems with an algebraic limit cycle of a given degree, preserving the degree of the algebraic limit cycle but increasing the degree of the system in one unity. To do this, we first list some results on the algebraic limit cycles of quadratic systems in the next two theorems.

Theorem 3 [6]. *After an affine change of variables, the unique quadratic systems having an algebraic limit cycle of degree 4 are as follows.*

(a) The Yablonskii's system

$$\begin{aligned}\dot{x} &= -4abcx - (a+b)y + 3(a+b)cx^2 + 4xy,\\ \dot{y} &= (a+b)abx - 4abcy + \left(4abc^2 - \tfrac{3}{2}(a+b)^2 + 4ab\right)x^2 + 8(a+b)cxy + 8y^2,\end{aligned} \tag{3}$$

with $abc \neq 0, a \neq b, ab > 0$ and $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$. This system possesses the irreducible invariant algebraic curve

$$(y + cx^2)^2 + x^2(x-a)(x-b) = 0 \tag{4}$$

of degree 4 having two components: an oval (the algebraic limit cycle) and an isolated point (a singular point).

(b) *The Filipstov's system*

$$\dot{x} = 6(1+a)x + 2y - 6(2+a)x^2 + 12xy,$$
$$\dot{y} = 15(1+a)y + 3a(1+a)x^2 - 2(9+5a)xy + 16y^2, \tag{5}$$

with $0 < a < 3/13$. *This system possesses the irreducible invariant algebraic curve*

$$3(1+a)(ax^2+y)^2 + 2y^2(2y - 3(1+a)x) = 0 \tag{6}$$

of degree 4 having two components, one is an oval and the other is homeomorphic to a straight line. This last component contains three singular points of the system.

(c) *The system*

$$\dot{x} = 5x + 6x^2 + 4(1+a)xy + ay^2, \qquad \dot{y} = x + 2y + 4xy + (2+3a)y^2, \tag{7}$$

with $(-71 + 17\sqrt{17})/32 < a < 0$ *possesses the irreducible invariant algebraic curve*

$$x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4 = 0 \tag{8}$$

of degree 4 having three components, one is an oval and each of the others is homeomorphic to a straight line. Each one of these last two components contains one singular point of the system.

(d) *The system*

$$\dot{x} = 2(1 + 2x - 2kx^2 + 6xy), \qquad \dot{y} = 8 - 3k - 14kx - 2kxy - 8y^2, \tag{9}$$

with $0 < k < 1/4$ *possesses the irreducible invariant algebraic curve*

$$\tfrac{1}{4} + x - x^2 + kx^3 + xy + x^2y^2 = 0 \tag{10}$$

of degree 4 having three components, one is an oval and each of the others is homeomorphic to a straight line. Each one of these last two components contains one singular point of the system.

Theorem 4. *[8]*

(a) *The system*

$$\dot{x} = 28x - \frac{12}{\alpha+4}y^2 - 2(\alpha^2-16)(12+\alpha)x^2 + 6(3\alpha-4)xy,$$
$$\dot{y} = (32-2\alpha^2)x + 8y - (\alpha+12)(\alpha^2-16)xy + (10\alpha-24)y^2, \tag{11}$$

has an irreducible invariant algebraic curve of degree 5 given by

$$x^2 + (16-\alpha^2)x^3 + (\alpha-2)x^2y + \frac{1}{(4+\alpha)^2}y^4 - \frac{6}{(4+\alpha)^2}y^5 - \frac{2}{4+\alpha}xy^2$$
$$+ \frac{(\alpha-4)(13+\alpha)}{4}x^2y^2 + \frac{12+\alpha}{4+\alpha}xy^4 + \frac{8-\alpha}{4+\alpha}xy^3 = 0. \tag{12}$$

For $\alpha \in (3\sqrt{7}/2, 4)$, *the curve (12) contains an algebraic limit cycle of degree 5.*

(b) *The system*

$$\dot{x} = 28(\beta-30)\beta x + y + 168\beta^2x^2 + 3xy,$$
$$\dot{y} = 16\beta(\beta-30)(14(\beta-30)\beta x + 5y + 84\beta^2x^2) + 24(17\beta-6)\beta xy + 6y^2, \tag{13}$$

has an irreducible invariant algebraic curve of degree 6 given by

$$\begin{aligned}
-7y^3 &+ 3(\beta-30)^2\beta y^2 + 18(\beta-30)(-2+\beta)\beta x y^2 + 27(\beta-2)^2\beta x^2 y^2\\
&+ 24(\beta-30)^3\beta^2 x y + 144(\beta-30)(\beta-2)^2\beta^2 x^3 y + 48(\beta-30)^4\beta^3 x^2\\
&+ 576(\beta-30)^2(\beta-2)^2\beta^3 x^4 - 432(\beta-2)^2\beta^2(3+2\beta)x^4 y\\
&- 3456(\beta-30)(-2+\beta)^2\beta^3(3+2\beta)x^5\\
&+ 3456(-2+\beta)^2\beta^3(12+\beta)(3+2\beta)x^6\\
&+ 24(\beta-30)^2\beta^2(9\beta-4)x^2 y + 64(\beta-30)^3\beta^3(9\beta-4)x^3 = 0. \qquad (14)
\end{aligned}$$

For $\beta \in (3/2, 2)$*, the curve (14) contains an algebraic limit cycle of degree 6.*

The following lemma is concerned with quadratic systems.

Lemma 5

(a) *[29] Let X and Y be functions of class* $\mathcal{C}^1$ *defined on a simply connected open region* $\mathcal{U}$ *of* $\mathbb{R}^2$*. If*

$$div(X(x,y),Y(x,y)) = \frac{\partial X(x,y)}{\partial x} + \frac{\partial Y(x,y)}{\partial y} = 0,$$

then the vector field

$$X(x,y)\frac{\partial x}{\partial} + Y(x,y)\frac{\partial}{\partial y}$$

has no limit cycles in $\mathcal{U}$*.*

(b) *[9, 29] For a quadratic system, there exists a unique singular point inside the bounded region limited by a closed orbit and it is either a center or a focus, and the determinant of the linear part at the singular point is nonzero.*

The next result provides examples of cubic systems having algebraic limit cycle of degree 4.

Proposition 6

(a) *The cubic system*

$$\begin{aligned}
\dot{x} &= x(-3(a+b)c + 4abcx - 4y + (a+b)xy),\\
\dot{y} &= -\tfrac{3}{2}a^2 + ab - \tfrac{3}{2}b^2 + 4abc^2 + ab(a+b)x + 5(a+b)cy + 4y^2 + (a+b)xy^2,
\end{aligned} \qquad (15)$$

has an irreducible invariant algebraic curve of degree 4 given by

$$1 + c^2 - (a+b)x + abx^2 + 2cxy + x^2y^2 = 0 \qquad (16)$$

with cofactor $4x(2abc + (a+b)y)$*. For* $abc \neq 0, a \neq b, ab > 0, 4c^2(a-b)^2 + (3a-b)(a-3b) < 0$*, the curve (16) contains an algebraic limit cycle of degree 4.*

(b) *The cubic system*

$$\begin{aligned}
\dot{x} &= -2x(-3(2+a) + 3(a+1)x + 6y + xy),\\
\dot{y} &= 3a(a+1) - 2(3+2a)y + 9(a+1)xy + 4y^2 - 2xy^2,
\end{aligned} \qquad (17)$$

has an irreducible invariant algebraic curve of degree 4 given by

$$3(a+1)(a+xy)^2 + 2xy^2(-3-3a+2y) = 0 \qquad (18)$$

with cofactor $2x(3+3a-4y)$*. For* $0 < a < 3/13$*, the curve (18) contains an algebraic limit cycle of degree 4.*

(c) *The cubic system*

$$\dot{x} = -x(6 + 5x + 4(a + 1)y + ay^2), \qquad \dot{y} = x - 2y - 3xy - (a + 2)y^2 - ay^3, \tag{19}$$

has an irreducible invariant algebraic curve of degree 4 given by

$$x + x^2 + xy + 2axy^2 + 2ay^3 + a^2y^4 = 0 \tag{20}$$

with cofactor $-2(3 + 5x + (2a + 3)y + 2ay^2)$. *For* $(-71 + 17\sqrt{17})/32 < a < 0$, *the curve (20) contains an algebraic limit cycle of degree 4.*

(d) *The cubic system*

$$\begin{aligned} \dot{x} &= 2x(2k - 2x - 6y - x^2)), \\ \dot{y} &= -14kx + 2ky + (8 - 3k)x^2 - 4xy - 20y^2 - 2x^2y, \end{aligned} \tag{21}$$

has an irreducible invariant algebraic curve of degree 4 given by

$$kx - x^2 + x^3 + \tfrac{1}{4}x^4 + x^2y + y^2 = 0 \tag{22}$$

with cofactor $4(k - 2x - 2x^2 - 10y)$. *For* $0 < k < 1/4$, *the curve (22) contains an algebraic limit cycle of degree 4.*

Proof. First of all, we prove that the algebraic limit cycles appearing in the quadratic systems (3), (5), (7) and (9) do not intersect the y-axis.

Substituting $x = 0$ into (4), we get $y^2 = 0$. This implies that the curve (4) intersects the y-axis at the unique point $(0, 0)$ which is a singular point of system (3). Hence, the algebraic limit cycle of system (3), contained in the curve (4), does not intersect the y-axis.

Using the same arguments as above, we conclude that the algebraic limit cycle of system (7) does not intersect the y-axis.

For system (5), we have

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=0} = 2y,$$

which implies that if a limit cycle of system (5) intersects the y-axis, then the origin is in the bounded region limited by this limit cycle. The origin is an unstable node, which contradicts the statement (b) of lemma 5.

A direct computation yields that the curve (10) and y-axis has no common point. Therefore, for all points (x, y) of an algebraic limit cycle appearing in the quadratic systems (3), (5), (7) and (9), we have $x \neq 0$. Applying the Poincaré transformation

$$x = \frac{1}{z}, \qquad y = \frac{u}{z}, \qquad \mathrm{d}t = z\,\mathrm{d}\tau \tag{23}$$

to the quadratic systems (3), (5), (7), (9) and replacing (z, u) again by (x, y), we get the systems (15), (17), (19), (21), respectively. The curves (16), (18), (20) and (22) are obtained from (4), (6), (8) and (10), respectively, by means of the same changes of coordinates and multiplication by z^4. □

The next two results provide examples of cubic systems having algebraic limit cycles of degrees 5 and 6.

Proposition 7. *The cubic system*

$$\begin{aligned} \dot{x} &= x\left(2(\alpha^2 - 16)(\alpha + 12) - 28x - 6(3\alpha - 4)y + \frac{12}{\alpha + 4}y^2\right), \\ \dot{y} &= -2(\alpha^2 - 16)x + (\alpha^2 - 16)(\alpha + 12)y - 20xy - 8\alpha y^2 + \frac{12}{\alpha + 4}y^3, \end{aligned} \tag{24}$$

has an irreducible invariant algebraic curve of degree 5 given by

$$-(\alpha^2-16)x^2+x^3+(\alpha-2)x^2y+\frac{1}{4}(\alpha-4)(\alpha+12)xy^2-\frac{2x^2y^2}{\alpha+4}$$
$$-\frac{(\alpha-8)xy^3}{\alpha+4}+\frac{(\alpha+12)y^4}{\alpha+4}+\frac{xy^4}{(\alpha+4)^2}-\frac{6y^5}{(\alpha+4)^2}=0 \qquad (25)$$

with cofactor $4(\alpha^2-16)(\alpha+12)-84x-2(19\alpha-12)y+\frac{60}{\alpha+4}y^2$*. For* $\alpha\in(3\sqrt{7}/2,4)$*, the curve (25) contains an algebraic limit cycle of degree 5.*

Proof. Since $(0, 0)$ is a singular point of system (11) and

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=0}=-\frac{12}{\alpha+4}y^2\leqslant 0,$$

the limit cycle of system (11) does not intersect the y-axis. Applying the Poincaré transformation (23) to the quadratic system (11), the claim of this proposition follows. □

Proposition 8. *The cubic system*

$$\dot{x}=-x\left(168\beta^2-84\beta(\beta+10)x+3y+\tfrac{28}{3}\beta(\beta+30)x^2\right),$$
$$\dot{y}=1344(\beta-30)\beta^3-672(\beta-30)\beta^2(\beta+10)x+48\beta(5\beta-3)y$$
$$+\tfrac{224}{3}\beta^2(\beta^2-900)x^2+28\beta(\beta-54)xy+3y^2-\tfrac{28}{3}\beta(\beta+30)x^2y, \qquad (26)$$

has an irreducible invariant algebraic curve of degree 6 given by

$$3456(-2+\beta)^2\beta^3(12+\beta)(3+2\beta)-10368(-2+\beta)^3\beta^3(3+2\beta)x$$
$$+576(-2+\beta)^2\beta^3(360-360\beta+41\beta^2)x^2-448\beta^3(-11520+30600\beta$$
$$-676\beta^2-442\beta^3+29\beta^4)x^3+112\beta^3(294480+86880\beta-6552\beta^2$$
$$-440\beta^3+33\beta^4)x^4-\frac{1568}{3}\beta^3(45648+3168\beta-440\beta^2-8\beta^3+\beta^4)x^5$$
$$+\frac{784}{27}\beta^3(140688-680\beta^2+\beta^4)x^6-432(-2+\beta)^2\beta^2(3+2\beta)xy$$
$$+1296(-2+\beta)^3\beta^2x^2y-168\beta^2(432-1092\beta+40\beta^2+3\beta^3)x^3y$$
$$+56\beta^2(-10632-1124\beta+82\beta^2+\beta^3)x^4y-\frac{1568}{3}\beta^2(\beta^2-396)x^5y$$
$$+27(\beta-2)^2\beta x^2y^2-504(\beta-2)\beta x^3y^2+2352\beta x^4y^2-7x^3y^3=0 \qquad (27)$$

with cofactor $-56\beta x(-6\beta+30x+\beta x)$*. For* $\beta\in(3/2,2)$*, the curve (27) contains an algebraic limit cycle of degree 6.*

Proof. For system (13), we have

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=-\frac{1}{3}}=\frac{28}{3}\beta(30+\beta)\neq 0,$$

which shows that the limit cycle of system (13) does not intersect the straight line $x=-1/3$. Applying the Poincaré transformation

$$z=\frac{1}{x+\frac{1}{3}},\qquad u=\frac{y}{x+\frac{1}{3}},\qquad \mathrm{d}t=z\,\mathrm{d}\tau$$

to the quadratic system (13) and replacing (z, u) again by (x, y), we get system (26). The curve (27) is obtained from (14) by means of the same changes of coordinates and multiplication by z^6. □

3. Ovals of degree 3

Lemma 9 [17]. *Any cubic Hamiltonian $H(x, y)$ such that its quadratic Hamiltonian system has a singular point of center type at the origin can be written via an affine change of variables into the form*

$$H(x, y) = \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{3}x^3 + axy^2 + \tfrac{1}{3}by^3, \tag{28}$$

where the parameters a and b are in the set

$$\Omega = \left\{-\tfrac{1}{2} \leqslant a \leqslant 1, 0 \leqslant b \leqslant (1-a)(1+2a)^{\frac{1}{2}}\right\}. \tag{29}$$

The oval of $H(x, y) = h$ around the center at the origin exists for the Hamiltonian values $h \in \Sigma = (0, 1/6)$.

Proposition 10. *The algebraic curve of degree 3 contains an oval without a point whose coordinates satisfy $f(x, y) = \partial f(x, y)/\partial x = \partial f(x, y)/\partial y = 0$ if and only if it can be written via an affine change of variables into the form*

$$H(x, y) = h, (a, b) \in \Omega, h \in \Sigma, \tag{30}$$

where H, Ω and Σ are defined as in lemma 9. Moreover, the algebraic curve (30)

(a) contains a unique oval and
(b) is irreducible if and only if $(b, h) \neq (0, (1 + 3a)/(24a^3))$.

Proof. By lemma 9, the 'if part' follows. For proving the 'only if part', we suppose that $f(x, y) = 0$ is an algebraic curve of degree 3 having an oval without point whose coordinates satisfy $f(x, y) = \partial f(x, y)/\partial x = \partial f(x, y)/\partial y = 0$. Then, $f(x, y) = 0$ is a trajectory of the quadratic Hamiltonian system

$$\dot{x} = \frac{\partial f(x, y)}{\partial y}, \qquad \dot{y} = -\frac{\partial f(x, y)}{\partial x}. \tag{31}$$

By lemma 5, the closed component of $f(x, y) = 0$ is not a limit cycle of system (31), which implies that it is a closed orbit surrounding a center and the determinant of linear part at this center is nonzero. By lemma 9, $f(x, y)$ can be written into the form (30). This completes the proof of the 'only if part'.

Assume that the algebraic curve $H(x, y) = h$ contains two ovals, which are two closed orbits of system

$$\dot{x} = \frac{\partial H}{\partial y} = y + 2axy + by^2, \qquad \dot{y} = -\frac{\partial H}{\partial x} = -x + x^2 - ay^2. \tag{32}$$

A picture with the phase portraits of this system in function of the parameters a and b can be found in figure 1 of [17]. By lemma 5 there exists a center inside each oval. If these two closed orbits surround the same center, then the straight line passing through the center should intersect in at least four points the algebraic curve $H(x, y) = h$ of degree 3, which is a contradiction. If these two closed orbits surround two different centers, then the straight line connecting the two centers should intersect in at least four points the algebraic curve $H(x, y) = h$ of degree 3, which again is a contradiction.

Suppose that $H(x, y) - h$ is reducible and $h \in \Sigma = (0, 1/6)$, then $H(x, y) - h$ can be written as the form

$$H(x, y) - h = (\alpha x + \beta y + \delta)H_2(x, y),$$

where α, β and δ are real constants, $\alpha^2 + \beta^2 \neq 0$, $H_2(x, y)$ is a polynomial of degree 2. Lemma 9 implies that the straight line $\alpha x+\beta y+\delta = 0$ does not intersect the oval $H_2(x, y) = 0$, which gives $\delta \neq 0$. Since $\alpha x + \beta y + \delta = 0$ is an invariant line of system (32), we have that $\alpha\dot{x} + \beta\dot{y} \equiv 0$ for the points (x, y) satisfying $\alpha x + \beta y + \delta = 0$. If $\beta \neq 0, \alpha \neq 0$, then

$$(\alpha\dot{x} + \beta\dot{y})|_{y=-\frac{\alpha x+\delta}{\beta}} = \frac{1}{\beta^2}(-\delta(\alpha\beta - b\alpha\delta + a\beta\delta) + (-\alpha^2\beta - \beta^3 + 2b\alpha^2\delta - 4a\alpha\beta\delta)x$$
$$+ (b\alpha^3 - 3a\alpha^2\beta + \beta^3)x^2) \equiv 0,$$

which yields

$$a = \frac{1}{2}\left(-1 + \frac{\beta^2}{\delta^2}\right), \qquad b = \frac{\beta(-\beta^2 + 3\delta^2)}{2\delta^3}, \qquad \alpha = -\delta. \tag{33}$$

If (33) holds, then

$$H(x, y) - h - (-\delta x + \beta y + \delta)H_2(x, y) = \tfrac{1}{6} - h,$$

where

$$H_2(x, y) = \frac{1}{6\delta^3}(-\delta^2 - \delta^2 x + 2\delta^2 x^2 + \beta\delta y + 2\beta\delta x y + (3\delta^2 - \beta^2)y^2).$$

This means that $H(x, y) - h$ is reducible if and only if $h = 1/6 \notin \Sigma$. If $\beta \neq 0, \alpha = 0$, then

$$(\alpha\dot{x} + \beta\dot{y})|_{y=-\frac{\delta}{\beta}} = \frac{1}{\beta}(-a\delta^2 - \beta^2 x + \beta^2 x^2) \not\equiv 0,$$

which is a contradiction.

If $\beta = 0$, then for the system (32),

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=-\frac{\delta}{\alpha}} = \frac{y(\alpha - 2a\delta + b\alpha y)}{\alpha} \equiv 0$$

if and only if $\alpha = 2a\delta, b = 0$, which implies

$$H(x, y) - h - \frac{1}{24a^3}(2ax + 1)(-1 - 3a + 2a(1 + 3a)x - 4a^2x^2 + 12a^3y^2) = \frac{1 + 3a}{24a^3} - h.$$

This yields that $H(x, y) - h$ is reducible if and only if $b = 0, h = (1 + 3a)/(24a^3)$. □

4. The algebraic limit cycles of degree ⩾ 3

In this section, we give some results about the algebraic limit cycles of degree $k \geqslant 3$.

A polynomial $f(x, y) \in \mathbb{C}[x, y]$ of degree k is said to be *regular at infinity* if the principal homogeneous part $\hat{f}$ of f (a homogeneous polynomial of degree k) is a product of k pairwise different linear forms, see [24] for more details.

If an irreducible algebraic curve $f(x, y) = 0$ of degree k is regular at infinity and contains a closed orbit (a limit cycle, or a period orbit surrounding a center) of a polynomial system of differential equations, then the closed orbit is called an *algebraic closed orbit regular at infinity of degree k*.

Lemma 11. *[7] Let*

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{34}$$

be a polynomial system of differential equations of degree j having an invariant algebraic curve $f(x, y) = 0$ of degree k. Assume that

(i) there are no points at which f and their first derivatives vanish simultaneously, and that
(ii) the highest order homogenous part of f has no repeated factors.

If $(f_x, f_y) = 1$*, then system (34) has the form*

$$x' = A(x, y) f(x, y) - D(x, y) f_y(x, y), \qquad y' = B(x, y) f(x, y) + D(x, y) f_x(x, y),$$

where $A(x, y)$, $B(x, y)$ *and* $D(x, y)$ *are polynomials with* $\deg A(x, y) \leqslant j-k$, $\deg B(x, y) \leqslant j-k$ *and* $\deg D(x, y) \leqslant j-k+1$.

We note that condition (ii) in lemma 11 means that $f(x, y)$ is regular at infinity.

Theorem 12. *A cubic system has an irreducible invariant algebraic curve containing an oval regular at infinity of degree 3 if and only if this system can be written via an affine change of variables into the form*

$$\begin{aligned} \dot{x} &= -(l + mx + ny) H_y(x, y) + r(H(x, y) - h) = P(x, y), \\ \dot{y} &= (l + mx + ny) H_x(x, y) + s(H(x, y) - h) = Q(x, y). \end{aligned} \tag{35}$$

Here, l, m, n, r *and* s *are real constants,* $l^2 + m^2 + n^2 \neq 0$, $H(x, y)$ *is defined in (28),* $h \in \Sigma = (0, 1/6)$, $b^2 - 4a^3 \neq 0$, $(b, h) \neq (0, (1 + 3a)/(24a^3))$ *and the system*

$$H(x, y) = h, \qquad l + mx + ny = 0 \tag{36}$$

has no solutions for $x \in (x_1(h), x_2(h))$*, where* $(x_1(h), 0)$ *and* $(x_2(h), 0)$ *are the two intersection points of the closed component of* $H(x, y) = h$ *with the x-axis. Moreover,*

(a) the algebraic closed orbit is a hyperbolic limit cycle if and only if $mr + ns \neq 0$ *and*
(b) if $mr + ns = 0$ *then system (35) has no limit cycles.*

Proof. If $(H_x, H_y) \neq 1$, then there exists a curve on which each point is a singular point of the Hamiltonian system (32). However, this phase portrait does not appear in figure 1 of [17]. Therefore, we have $(H_y, H_x) = 1$. On the other hand, the curve $H(x, y) = h$ is regular at infinity if and only if $b^2 - 4a^3 \neq 0$. Proposition 10 shows that $H(x, y) = h$ is irreducible if and only if $(b, h) \neq (0, (1 + 3a)/(24a^3))$.

Proposition 10 and lemma 11 imply that a cubic system with an algebraic closed curve of degree 3 can be written via an affine change of variables into the form (35). Of course, this closed curve may be a limit cycle, a homoclinic loop, a heteroclinic loop or a periodic orbit surrounding a center.

Let Γ_h be the oval of the algebraic curve $H(x, y) = h$. Then, Γ_h is a periodic orbit of system (35) if and only if there does not exist any singular point on Γ_h. There are two types of singular points on $H(x, y) = h$, those whose coordinates satisfy

$$H(x, y) = h, \qquad H_y(x, y) = 0, \qquad H_x(x, y) = 0, \tag{37}$$

and those whose coordinates satisfy (36). By proposition 10, singular points satisfying (37) do not exist. Hence, (36) has no solution in the interval $(x_1(h), x_2(h))$ if and only if Γ_h is a periodic orbit of system (35).

It is well known (see for instance [10]) that the periodic orbit Γ_h is a hyperbolic limit cycle if and only if

$$\int_0^{T(h)} \operatorname{div}(P, Q)\, \mathrm{d}t = \int_0^{T(h)} \left(\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \right) \mathrm{d}t \neq 0,$$

where $T(h)$ is the period of Γ_h. Since $H(x, y) = h$ is an invariant algebraic curve, we have on it that $H_x(x, y)\, \mathrm{d}x + H_y(x, y)\, \mathrm{d}y = 0$. On the other hand, from system (35) we obtain

$$\mathrm{d}t = \frac{\mathrm{d}x}{P(x, y)}.$$

Therefore,

$$\begin{aligned}\int_0^{T(h)} \mathrm{div}(P,Q)\,\mathrm{d}t &= \oint_{\Gamma_h}\left(\frac{\partial P(x,y)}{\partial x}+\frac{\partial Q(x,y)}{\partial y}\right)\frac{\mathrm{d}x}{P(x,y)}\\ &= \oint_{\Gamma_h}\frac{(r+n)H_x(x,y)+(s-m)H_y(x,y)}{-(l+mx+ny)H_y(x,y)}\,\mathrm{d}x\\ &= \oint_{\Gamma_h}\frac{m-s}{l+mx+ny}\,\mathrm{d}x-\oint_{\Gamma_{\bar h}}\frac{(r+n)H_x(x,y)}{(l+mx+ny)H_y(x,y)}\,\mathrm{d}x\\ &= \oint_{\Gamma_h}\frac{m-s}{l+mx+ny}\,\mathrm{d}x+\oint_{\Gamma_{\bar h}}\frac{r+n}{(l+mx+ny)}\,\mathrm{d}y\\ &= \iint_{Int\Gamma_h}\frac{mr+ns}{(l+mx+ny)^2}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we use the Green formula and suppose that Γ_h has the counterclockwise orientation. Therefore, the period orbit Γ_h is a hyperbolic limit cycle if and only if $mr+ns\neq 0$. So statement (a) is proved.

If $mr+ns = 0, m^2+n^2\neq 0$, then $l+mx+ny = 0$ is an invariant algebraic curve with cofactor $K_1(x,y) = nH_x(x,y)-mH_y(x,y)$. On the other hand, the cofactor of the invariant algebraic curve $H(x,y) = h$ is $K_2(x,y) = rH_x(x,y)+sH_y(x,y)$. Therefore, $sK_1(x,y)+mK_2(x,y)\equiv 0$. It follows from Darboux theorem (see, for instance, theorem 14 in [6]) that system (35) has a first integral of the form

$$(l+mx+ny)^s(H(x,y)-h)^m. \tag{38}$$

If $m = n = 0, l\neq 0$, then the system (35) has a first integral of the form

$$l\ln|H(x,y)-h|+sx-ry. \tag{39}$$

From (38) and (39) statement (b) follows easily. □

Proposition 13. *If $r\neq 0, (a,b)\in\Omega, h\in\Sigma, l\in\left(-\infty,-\frac{1}{2}\right)\cup(1,+\infty)$ and $(b,h)\neq(0,(1+3a)/(24a^3))$, then the system*

$$\begin{aligned}\dot x &= -(x-l)H_y(x,y)+r(H(x,y)-h) = \bar P(x,y),\\ \dot y &= (x-l)H_x(x,y)+s(H(x,y)-h) = \bar Q(x,y),\end{aligned}\tag{40}$$

has an irreducible hyperbolic algebraic limit cycle of degree 3, where $H(x,y), \Omega, \Sigma$ are defined as in lemma 9.

Proof. We first prove that there is no singular point on the oval Γ_h of algebraic curve $H(x,y) = h$. There are two types of critical points on $H(x,y) = h$: those whose coordinates satisfy (37), and those whose coordinates satisfy

$$H(x,y) = h,\qquad x = l. \tag{41}$$

We know that singular points satisfying (37) do not exist by proposition 10.

Denoted by $\Gamma_{1/6}$, the homoclinic loop of the quadratic Hamiltonian system (32) is defined by $H(x,y) = \frac{1}{6}$. By direct computation we know that $\Gamma_{1/6}$ intersects the x-axis at the points $(1,0)$ and $\left(-\frac{1}{2},0\right)$, which implies that the line $x = l$ does not intersect $\Gamma_{1/6}$. Since in the closed orbit Γ_h of system (32) lies the bounded region limited by the homoclinic loop $\Gamma_{1/6}$, it does not intersect the straight line $x = l$. Therefore, there is no singular point on the oval Γ_h of algebraic curve $H(x,y) = h$ for $h\in\Sigma$, which shows that Γ_h is a periodic orbit.

By the same arguments as in the proof of theorem 12, we obtain that

$$\int_0^{T(h)}\mathrm{div}(\bar P,\bar Q)\,\mathrm{d}t = \oint_{\Gamma_h}\frac{1-s}{x-l}\,\mathrm{d}x+\frac{r}{x-l}\,\mathrm{d}y = -\mathrm{sgn}(l)\iint_{Int\Gamma_h}\frac{r}{(x-l)^2}\,\mathrm{d}x\,\mathrm{d}y\neq 0,$$

where $T(h)$ is the period of the periodic orbit Γ_h. Consequently, Γ_h is an irreducible hyperbolic algebraic limit cycle of degree 3. □

In the rest of this section, we consider algebraic limit cycles of polynomial systems of differential equations of degree $i \geqslant 3$. First, we prove the following theorem.

Theorem 14. *Any oval of an irreducible algebraic curve $f(x, y) = 0$ of degree k without a point (x, y) satisfying $\partial f(x, y)/\partial x = \partial f(x, y)/\partial y = 0$ is an algebraic limit cycle of a convenient polynomial system of differential equations of degree i, $i \geqslant k+1$.*

Proof. If algebraic curve $f(x, y) = 0$ of degree k contains an oval without a point (x, y) satisfying $\partial f(x, y)/\partial x = \partial f(x, y)/\partial y = 0$, then it is a closed orbit of the Hamilton system $\dot{x} = \partial f(x, y)/\partial y, \dot{y} = -\partial f(x, y)/\partial x$. Let $f(x, y) = h$ be a first integral of this Hamiltonian system and (h_c, h_s) be the maximum interval of existence of the closed orbits $\Gamma_h \subset \{(x, y)|f(x, y) = h\}, h_c < 0 < h_s$. Consider the perturbed polynomial system of degree $i \geqslant k+1$:

$$\dot{x} = \frac{\partial f(x, y)}{\partial y}, \qquad \dot{y} = -\frac{\partial f(x, y)}{\partial x} + \varepsilon f(x, y)(y + x^{i-k}). \tag{42}$$

It follows by direct computation that $f(x, y) = 0$ is an invariant algebraic curve of system (42) with cofactor $K(x, y) = \varepsilon f_y(x, y)(y + x^{i-k})$. The Abelian integral, associated with system (42), is given by

$$I(h) = \oint_{\Gamma_h} f(x, y)(y + x^{i-k})\, \mathrm{d}x = hI_0(h), \tag{43}$$

where we suppose that the oval Γ_h has a clockwise orientation and

$$I_0(h) = \oint_{\Gamma_h} y\, \mathrm{d}x = \iint_{Int\Gamma_h} \mathrm{d}x\, \mathrm{d}y \neq 0. \tag{44}$$

It is well known (see, for instance, [19] or [1] p 313) that the displacement function of the perturbed system (42) can be expressed in the form

$$d(h, \varepsilon) = \varepsilon I(h) + O(\varepsilon^2), \tag{45}$$

and the following statements hold when $I(h) \not\equiv 0$.

(a) If there exists $h^* \in (h_c, h_s)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$, then system (42) has a unique limit cycle bifurcating from Γ_{h^*}, moreover, this limit cycle is hyperbolic.
(b) The total number (counting the multiplicities) of the limit cycles of system (42) bifurcation from the period annulus of the Hamiltonian system (42) with $\varepsilon = 0$ is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of the Abelian integral $I(h)$ for $h \in (h_c, h_s)$.
(c) $I(h)$ is an analytic function in $h \in (h_c, h_s)$.

We note that $I(h)$ has a unique zero at $h = 0$ and $I'(0) = I_0(0) \neq 0$. Since (43) and (44) implies $d(h, \varepsilon) \not\equiv 0$, the oval contained in $f(x, y) = 0$ is a hyperbolic algebraic limit cycle. □

Now we give the main results of this section.

Theorem 15. *Any oval of an irreducible algebraic curve $f(x, y) = 0$ of degree 3 without a point (x, y) satisfying $\partial f(x, y)/\partial x = \partial f(x, y)/\partial y = 0$ is an algebraic limit cycle of a convenient polynomial system of differential equations of degree i, $i \geqslant 3$.*

Proof. The statement follows from proposition 13 and theorem 14. □

Recall (see the introduction) that quadratic systems cannot have algebraic limit cycles of degree 3, but theorem 15 shows that for arbitrary degree > 2 there are polynomial systems of differential equations having algebraic limit cycles of degree 3.

Remark 16. Suppose that the algebraic curve $f(x, y) = 0$ of degree k contains an oval. Instead of system (42), consider the perturbed system of degree $i \geqslant kj + 1$:

$$\dot{x} = \frac{\partial f}{\partial y}, \qquad \dot{y} = -\frac{\partial f}{\partial x} + \varepsilon(f - h_1)(f - h_2)\cdots(f - h_j)g(x, y), \tag{46}$$

where $g(x, y)$ is a polynomial of degree $i - kj$ and $h_l \in (h_c, h_s), l = 1, 2, \ldots, j$. By direct computation, $f(x, y) = h_l$ is an invariant algebraic curve with cofactor $K_l(x, y) = \varepsilon f_y(f - h_1)\cdots(f - h_{l-1})(f - h_{l+1})\cdots(f - h_j)g(x, y)$. The associated Abelian integral is given by

$$I(h) = (h - h_1)(h - h_2)\cdots(h - h_j)\oint_{\Gamma_h} g(x, y)\,\mathrm{d}x.$$

(a) If $g(x, y) = y + x^{i-kj}$, then system (46) has j algebraic limit cycles of degree k. Moreover, these limit cycles are hyperbolic if $h_{l_1} \neq h_{l_2}$ for $l_1 \neq l_2$.

(b) If $\oint_{\Gamma_h} g(x, y)\,\mathrm{d}x$ has v simple zeros different from h_l for $l = 1, 2, \ldots, j$, then system (46) has at least $v + j$ limit cycles. Note that these additional v limit cycles in general are not algebraic.

5. Invariant algebraic homoclinic loop of degree 3

It is easy to find examples of period annulus whose boundary is given by an algebraic homoclinic loop of degree 3. In this section, we only consider for cubic systems algebraic homoclinic loops which are not boundary of a period annulus.

Recall that $X = \bar{X}$ is called a *hyperbolic* singular point of system $\dot{X} = \mathcal{R}(X)$, $X \in \mathbb{R}^n$ if $\mathcal{R}(\bar{X}) = 0$, and none of the eigenvalues of $D\mathcal{R}(\bar{X})$ has zero real part [27].

Theorem 17. *If an irreducible algebraic curve regular at infinity of degree 3 contains a homoclinic loop of a cubic system with a hyperbolic saddle, then it can be reduced to* $H(x, y) = \frac{1}{6}$, *where* $H(x, y)$ *is defined in (28),* $b^2 - 4a^3 \neq 0, b \neq (1 - a)(1 + 2a)^{1/2}(a, b) \in \Omega$.

Proof. Suppose that the algebraic curve $f(x, y) = 0$ regular at infinity of degree 3 contains a homoclinic loop of a cubic system with a hyperbolic saddle (x_0, y_0), then either it can be put into the form (30), or the closed component of $f(x, y) = 0$ contains the singular point (x_0, y_0) satisfying $\partial f(x_0, y_0)/\partial x = \partial f(x_0, y_0)/\partial y = 0$.

If the algebraic curve is reduced to the normal form (30), then it follows from lemma 11 that this cubic system can be written in the form of system (35), where l, m, n, r and s are also real constants, $l^2 + m^2 + n^2 \neq 0$, $b^2 - 4a^3 \neq 0$, $H(x, y) = h$ contains a homoclinic loop of system (35), denoted by $\bar{\Gamma}_h$. Note that we do not need other conditions in theorem 12 except ones given explicitly.

Let (x_0, y_0) be the hyperbolic saddle on $\bar{\Gamma}_h$. Then (x_0, y_0) satisfies either (36), or (37). It follows from proposition 10 that (x_0, y_0) is not a solution of (37). Therefore, assume (x_0, y_0) satisfies (36).

Since $\bar{\Gamma}_h$ is a homoclinic loop, the straight line $l+mx+ny=0$ contacts $\bar{\Gamma}_h$ at an unique point (x_0, y_0). This implies that one of the following two equations has exactly one solution with multiplicity 2:

$$H\left(x, -\frac{l+mx}{n}\right) = h, \qquad n \neq 0,$$
$$H\left(-\frac{l}{m}, y\right) = h, \qquad n = 0, \quad m \neq 0,$$

which yields

$$nH_x(x_0, y_0) - mH_y(x_0, y_0) = 0, \qquad n \neq 0, \tag{47}$$

$$H_y(x_0, y_0) = 0, \qquad n = 0, \qquad m \neq 0, \qquad x_0 = -\frac{l}{m}. \tag{48}$$

If (47) holds, then

$$\det\begin{pmatrix} \frac{\partial P(x,y)}{\partial x} & \frac{\partial P(x,y)}{\partial y} \\ \frac{\partial Q(x,y)}{\partial x} & \frac{\partial Q(x,y)}{\partial y} \end{pmatrix}_{(x_0,y_0)}$$
$$= \det\begin{pmatrix} -mH_y(x_0, y_0) + rH_x(x_0, y_0) & (-n+r)H_y(x_0, y_0) \\ (m+s)H_x(x_0, y_0) & nH_x(x_0, y_0) + sH_y(x_0, y_0) \end{pmatrix}$$
$$= (m+s)(r-n)\det\begin{pmatrix} H_x(x_0, y_0) & H_y(x_0, y_0) \\ H_x(x_0, y_0) & H_y(x_0, y_0) \end{pmatrix} = 0.$$

This means that the singular point (x_0, y_0) is not a hyperbolic saddle point. If (48) holds, then (x_0, y_0) is also not a hyperbolic saddle by the same arguments as above. Therefore, if the algebraic curve $f(x, y) = 0$ regular at infinity of degree 3 contains a homoclinic loop with hyperbolic saddle (x_0, y_0), then $\partial f(x_0, y_0)/\partial x = \partial f(x_0, y_0)/\partial y = 0$.

The algebraic $f(x, y) = 0$ should be a trajectory of the quadratic Hamiltonian system (31). In [2], the authors proved that there are exactly 28 non-equivalent topological phase portraits of quadratic Hamiltonian systems. The phase portraits appearing in [2] show that if the quadratic Hamiltonian system has a homoclinic loop, then there is a center inside this loop. Hence, it follows from lemma 9 that $f(x, y) = 0$ can be put into the normal form $H(x, y) = 1/6$. Finally, it follows from the proof of proposition 10 (in particular, see (33)) that $H(x, y) = 1/6$ is reducible if and only if $b = (1-a)(1+2a)^{1/2}$. The statement of the theorem is proved. □

Finally, we show that $H(x, y) = 1/6$ is a homoclinic loop of a convenient cubic system. The definition of the hyperbolic stability of a homoclinic loop is given by the integral of the divergence along the homoclinic orbit. This definition can be obtained passing to the limit the usual definition of a hyperbolic limit cycle, see for instance [10].

Proposition 18. *Suppose that $H(x, y)$ is defined in (28), $b \neq (1-a)(1+2a)^{1/2}(a, b) \in \Omega, l \in \left(-\infty, -\frac{1}{2}\right) \cup (1, +\infty), r \neq 0$. Then the algebraic curve $H(x, y) = \frac{1}{6}$ of degree 3 contains an irreducible homoclinic loop of system*

$$\begin{aligned} \dot{x} &= -(x-l)H_y(x, y) + r\left(H(x, y) - \tfrac{1}{6}\right) = \widetilde{P}(x, y), \\ \dot{y} &= (x-l)H_x(x, y) + s\left(H(x, y) - \tfrac{1}{6}\right) = \widetilde{Q}(x, y). \end{aligned} \tag{49}$$

The homoclinic loop is hyperbolic stable (resp. unstable) if $\operatorname{sgn}(l)r > 0$ *(resp.* $\operatorname{sgn}(l)r < 0$*).*

Proof. By the same arguments as proposition 13, we conclude that the closed component $\Gamma_{1/6}$ of algebraic curve $H(x, y) = 1/6$ just contains a hyperbolic saddle $(1, 0)$, and

$$\int_{-\infty}^{\infty} \operatorname{div}(\widetilde{P}, \widetilde{Q})\, \mathrm{d}t = \oint_{\Gamma_{1/6}} \frac{1-s}{x-l}\, \mathrm{d}x + \frac{r}{x-l}\, \mathrm{d}y = -\operatorname{sgn}(l) \iint_{Int\Gamma_{1/6}} \frac{r}{(x-l)^2}\, \mathrm{d}x\, \mathrm{d}y \neq 0.$$

Therefore, $\Gamma_{1/6}$ is a stable (resp. unstable) homoclinic loop if $\operatorname{sgn}(l)r > 0$ (resp. $\operatorname{sgn}(l)r < 0$). □

6. Cubic systems with two algebraic limit cycles

In this section, we present a cubic system with two algebraic limit cycles surrounding two different foci, one is contained in $x > 0$ and the other in $x < 0$.

Proposition 19. *System*

$$\dot{x} = 2y(10 + xy), \qquad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3 \tag{50}$$

possesses two algebraic limit cycles contained in the invariant algebraic curve

$$\tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 = -\tfrac{1}{8}. \tag{51}$$

Proof. System (50) has three singular points in the finite plane: a saddle at $(0, 0)$ and two stable foci at $(\pm 1, 0)$. The Hamiltonian system

$$\dot{x} = y, \qquad \dot{y} = x - x^3$$

has a first integral $H(x, y) = y^2/2 - x^2/2 + x^4/4 = h$ which corresponds to two closed orbits surrounding the center $(\pm 1, 0)$ if $h \in (-1/4, 0)$. This yields that the algebraic curve (51) contains two ovals. It is easy to prove that (51) is an invariant algebraic curve of system (50) with cofactor $K(x, y) = 8y^2$. Since there are no singular points of system (50) on the curve (51) and the foci $(\pm 1, 0)$ are inside the two ovals, the statement follows. □

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